Mañé's Critical Value for Non-Compact Manifolds

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Setting

Let M be a smooth, connected Riemannian manifold without boundary, equipped with a complete Riemannian metric defining the norm $\|\cdot\|$. We assume that the *Lagrangian* $L : TM \to \mathbb{R}$ is a *Tonelli Lagrangian*; that is, it is C^2 and satisfies the following conditions:

(L1) for every $K \geq 0$, there exists $C(K) \in \mathbb{R}$ such that

$$L(x,v) \geq K\|v\| - C(K) \quad \text{for all } (x,v) \in TM;$$

(L2) for every $R \geq 0$, we have

$$A(R) := \sup\{L(x,v) : \|v\| \leq R\} < +\infty;$$

(L3) for every $(x,v) \in TM$, the second derivative along the fibers $\partial^2 L/\partial v^2(x,v)$ is strictly positive definite.

Given $x, y \in M$, $t > 0$, let $\mathrm{AC}(x,y;t)$ denote the set of absolutely continuous curves $\gamma : [0,t] \to M$ with $\gamma(0) = x, \gamma(t) = y$. We define the *cost* as

$$c_t^L(x,y) := \inf_{\gamma \in \mathrm{AC}(x,y;t)} \int_0^t L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s,$$

and the *Mañé potential* as

$$c^L(x,y) := \inf_{t>0} c_t^L(x,y).$$

Let us also define the *Euler-Lagrange flow* $f_t : TM \to TM$ by $f_t(w) := (\gamma_w(t), \dot\gamma_w(t))$, where $\gamma_w : \mathbb{R} \to M$ is the solution to the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v}(\gamma, \dot\gamma)\right) = \frac{\partial L}{\partial x}(\gamma, \dot\gamma) \qquad \text{(EL)}$$

with initial condition $(\gamma(0), \dot\gamma(0)) = w$. Properties of the Euler-Lagrange flow are closely linked to the geometry of M (cf., for example, [4, 2]).

Finally, we define the *Hamiltonian* $H : T^*M \to \mathbb{R}$ by

$$H(x,p) := \sup_{v \in T_xM} \langle p, v\rangle - L(x,v).$$

The Hamiltonian H satisfies similar properties to the Lagrangian L (cf. [7, Theorem 2.1]).

History

This setting has been considered in ergodic theory, weak KAM theory, and optimal transport (OT). Some relevant litterature is summarized below.

	Compact	Non-compact
Ergodic theory	Mañé [9]	?
Weak KAM theory	Fathi [5]	Fathi-Maderna [7]
OT for c_t^L	classical	Fathi-Figalli [6]
OT for c^L	Bernard-Buffoni [1]	Figalli [8]

Mañé's Critical Value

By computing c^{L+k} for $L(x,v) = \frac{1}{2}\|v\|^2$, $k \in \mathbb{R}$, one may guess the existence of some sort of *critical value* for Tonelli Lagrangians. This was proven by Mañé [9, Theorem I].

Theorem. *There exists $k_0 \in \mathbb{R}$ such that*

(a) if $k < k_0$, then $c^{L+k}(x_1,x_2) = -\infty$ for all $x_1, x_2 \in M$;

(b) if $k \geq k_0$, then $c^{L+k}(x_1,x_2) > -\infty$ for all $x_1, x_2 \in M$, c^{L+k} is Lipschitz, and c^{L+k} satisfies

$$c^{L+k}(x_1,x_3) \leq c^{L+k}(x_1,x_2) + c^{L+k}(x_2,x_3)$$

for all $x_1, x_2, x_3 \in M$; and

(c) if $k > k_0$, then the Lagrangian $L + k$ is supercritical*; that is,*

$$c^{L+k}(x_1,x_2) + c^{L+k}(x_2,x_1) > 0$$

for all $x_1, x_2 \in M$, $x_1 \neq x_2$.

When M is compact, two important facts about the critical value k_0 are known. First, it has an *ergodic determination* (cf. [9, Theorem II]):

$$k_0 = -\min_{\mu \in \mathcal{M}(L)} \int_{TM} L\,\mathrm{d}\mu. \qquad \text{(ED)}$$

Secondly, k_0 is the minimum value such that the Hamilton-Jacobi equation

$$H(x, \mathrm{d}_x u) = c, \qquad \text{(HJ)}$$

$c \in \mathbb{R}$, admits a global viscosity subsolution (that is, a suitably defined weak subsolution) $u : M \to \mathbb{R}$. This is known as the *weak KAM theorem* (cf. [5]).

Non-Compact Manifolds

In the non-compact setting, the weak KAM theorem is still known to hold (cf. [7]). Furthermore, many dynamical properties of the Euler-Lagrange flow (f_t) are equivalent to the existence of certain viscosity subsolutions of the Hamilton-Jacobi equation (HJ). For example, the existence of a compact $K \subset M$ and a viscosity subsolution $w : M \to \mathbb{R}$ of (HJ) such that

$$H(x, \mathrm{d}_x w) < c \quad \text{on } M \setminus K$$

is equivalent to the compactness of the *Aubry set* $\mathcal{A}(L)$.

In work in progress, we prove the following.

Theorem. *Assume that there exists a compact $K \subset M$, a $\delta > 0$, and a viscosity subsolution $w : M \to \mathbb{R}$ of* (HJ) *such that*

$$H(x, \mathrm{d}_x w) \leq c - \delta \quad \text{on } M \setminus K.$$

Then (ED) *holds.*

This assumption appears naturally, for example, when studying compactly supported perturbation of periodic Hamiltonians (cf. [3]). Many of Mañé's other results in [9] also generalize to this setting.

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