Thermodynamic Formalism for Local Bowen Potentials on Topological Markov Shifts

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Preliminaries

Let $\mathcal{A}$ be a countable set. A *shift space* X is a subset of the *full shift* $\mathcal{A}^{\mathbb{N}}$ which is closed in the product topology of discrete topologies on $\mathcal{A}$ and invariant under the *shift map* σ. X is complete and separable, and is compact if and only if $\mathcal{A}$ is finite.

Let $\mathcal{L}(X)$ ($\mathcal{L}$ for short) denote the *language* of X. If X is finite-state, we say that X has the *specification property* if there exists $\tau \in \mathbb{N}$ such that, for all $u, w \in \mathcal{L}$, there exists $v \in \mathcal{L}_\tau$ with $uvw \in \mathcal{L}$. Modern results on the specification property are surveyed in [2].

The most well-studied countable-state shift spaces are *topological Markov shifts* (*TMS*), which are generalized shifts of finite type. Given a topologically mixing TMS X, a potential $\varphi : X \to \mathbb{R}$, and $a \in \mathcal{A}$, we define the *Gurevich pressure* of φ to be

$$P_G(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\sigma^n x = x, x_0 = a} e^{S_n\varphi(x)},$$

where $S_n\varphi(x) := \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1}x)$. Let $\mathrm{var}_n(\varphi, \mathcal{L}) := \sup\{|\varphi(x) - \varphi(y)| : x, y \in [\omega], \omega \in \mathcal{L}_n\}$. The variational principle holds for potentials which satisfy the Walters property; that is, for all $k \geq 1$, $\mathrm{var}_{n+k}(S_n\varphi, \mathcal{L}) < \infty$ and $\mathrm{var}_{n+k}(S_n\varphi, \mathcal{L}) \to 0$ as $k \to \infty$. Further, the existence and uniqueness of equilibrium states can be classified using the eigenfunction and eigenmeasure of *Ruelle's transfer operator* $L_\varphi f(x) := \sum_{\sigma y = x} e^{\varphi(y)} f(y)$ and its dual and a generalized Ruelle's Perron-Frobenius Theorem. More information can be found in [3].

The Generalized Specification Property and the Bowen Property

Fix a countable-state shift space X. Given $I \subseteq \mathcal{A}$, let $\mathcal{L}^I := \{\omega \in \mathcal{L} : \omega_0 \in I, \omega a \in \mathcal{L} \text{ for some } a \in I\}$. We say that X satisfies the *generalized specification property* if, for all finite $I \subseteq \mathcal{A}$, there exists $\tau = \tau(I) \in \mathbb{N}$ such that, for all $u, w \in \mathcal{L}^I$, there exists $v \in \mathcal{L}_\tau$ with $uvw \in \mathcal{L}^I$. If X is a TMS, this property is equivalent to topological mixing. A large class of countable-state sofic shifts, as studied by Sobottka [4], serve as non-TMS examples.

Given a potential $\varphi : X \to \mathbb{R}$, we say that φ satisfies the *local Bowen property* if, for all finite $I \subseteq \mathcal{A}$, there exists $Q = Q(I) > 0$ such that, for all $n \in \mathbb{N}$, $\mathrm{var}_{n+1}(S_n\varphi, \mathcal{L}^I) < Q$. This property is "local" rather than "global" since we could have that $\mathrm{var}_{n+1}(S_n\varphi, \mathcal{L}) = \infty$ and is weaker than all of the regularity properties considered in [3].

Results

Theorem (P.-Somers-Thompson)

Let X be a countable-state shift space with specification and $\varphi : X \to \mathbb{R}$ be a potential with the local Bowen property. Then the variational principle holds.

Theorem (P.)

Let X be a (suspension flow over a) countable-state sofic shift with specification and $\varphi : X \to \mathbb{R}$ be a potential with the local Bowen property. Then a generalized Ruelle's Perron-Frobenius theorem holds.

Future Directions

- Can these improved results be used to extend the thermodynamic formalism for multidimensional piecewise expanding maps in [1]?
- Can transfer operator-theoretic methods be used to remove the local compactness assumption in the upcoming work of Climenhaga, Thompson, and Wang?
- Is there a generalized Ruelle's transfer operator that allows us to lift these results to general countable-state shift spaces with specification?

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References

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