ON ROBIN'S INEQUALITY AND THE KANEKO-LAGARIAS INEQUALITY

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Abstract. We prove that Robin's inequality and the Lagarias inequality hold for almost every number, including all numbers not divisible by one of the prime numbers 2, 3, or 5, primorials, sufficiently big numbers of the form $2^k n$ for odd n and 21-free integers. We also prove that the Kaneko-Lagarias inequality holds for all numbers if and only if it holds for all superabundant numbers.

Contents

1. Introduction

We define the *sum of divisors function* and *Euler's totient function* as

$$\sigma(n) = \sum_{d|n} d, \quad \varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

respectively, where the product is taken over primes p which divide n. It has long been known that these functions have connections to the Riemann hypothesis (RH). Robin's inequality [Rob84] states that RH holds if and only if, for all $n > 5040$,

$$\sigma(n) < e^{\gamma} \log(\log(n)), \tag{1.1}$$

where $\gamma \approx .57721\ldots$ denotes the Euler-Mascheroni constant.

Let us briefly survey some known results concerning families of natural numbers which satisfy Robin's inequality. A number is k-free if all the powers in its prime factorization are

Date: August 11, 2025.
2020 *Mathematics Subject Classification.* 11N56, 11M26.

$< k$. In [Cho+07], it is proven that all all odd integers > 9 satisfy Robin's inequality, as do all 5-free integers. This was extended to 7-free numbers in [SP12], 11-free numbers in [BT15], 20-free numbers in [MP20], and finally 21-free numbers in [Axl23]. We reprove the result of Axler [Axl23, Theorem 1] using more elementary methods in Theorem 2.16: while Axler's proof relies on estimates from [SP12] and combinatorial prime counting algorithms, our proof uses only arithmetic manipulations and a sharper bound from [AN22] than the one used in [RS62, Theorem 15].

Similarly, Theorem 2.12, which states that Robin's inequality holds for numbers not divisible by one of the primes 2, 3, or 5, is implied by the results of [Her16], but is proven using more elementary methods. The proof in [Her16], which works with a number's *p-adic order*, relies on an algorithm from [AFJ07], whereas ours is derived only from arithmetic manipulations.

The final class of results concerning Robin's inequality is density results. The first such result is by Robin [Rob84], who proved that Robin's inequality holds for all squarefree (that is, 2-free) numbers. In [Ten95, p. 46], it is shown that the logarithmic density of non-squarefree integers is $\frac{1}{2} - \frac{2}{\pi^2} \approx .2973....$ Similarly, Theorem 2.12 shows that Robin's inequality holds for a set of logarithmic density $\frac{29}{30}$. Wójtowicz [Wój07] was the first to show that Robin's inequality holds on a set of density 1. We prove this using different methods in Theorem 2.18: our proof, again, relies mostly on arithmetic manipulations, as opposed to the "deep" results of Ford and Luca-Pomerance.

Another well-studied inequality which is equivalent to RH is the Lagarias inequality. Denote by H_n the n-th *harmonic number*; that is, $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. The Lagarias inequality [Lag02] states that, for all $n \geq 1$,

$$\sigma(n) < H_n + \exp(H_n)\log(H_n).$$

It turns out that the H_n term on the RHS is negligible in the sense that the following inequality, which we name the Kaneko-Lagarias inequality (see the acknowledgements in [Lag02]), is also equivalent to RH: for all $n > 60$,

$$\sigma(n) < \exp(H_n)\log(H_n).$$

We note that similar alternative inequalities have been introduced [WY21].

A number is *superabundant* if $m < n$ implies $\sigma(m)/m < \sigma(n)/n$. By dividing both sides of (1.1) by n and noting that the RHS is monotone increasing, one immediately sees that Robin's inequality holds if and only if it holds for superabundant numbers (this observation was made in [AF09]). One would like to say the same for the Lagarias inequality and the Kaneko-Lagarias inequality, but the picture is more complicated since monotonicity is harder to prove. Nevertheless, we prove in Theorem 3.9 that the Kaneko-Lagarias inequality holds if and only if it holds for superabundant numbers. We would like to extend this result to the Lagarias inequality in future work.

The layout of our paper is as follows. In Section 2, we consider Robin's inequality. Each subsection corresponds to a result which we prove, and is labeled as such. We note that we use a unified method throughout the section, reobtaining some results which were found using varied methods. In Section 3, we focus on the Lagarias inequality. We introduce the Kaneko-Lagarias inequality and prove the contents of the last paragraph.

2. Robin's Inequality

2.1. **Sufficiently big numbers not divisible by one of the prime numbers 2,3,5.** Let $p_1 = 2$, $p_2 = 3$, etc. be an enumeration of the prime numbers which we denote by $\mathbb{P}$.

Fix $j \in \mathbb{N}$ and let $q_1 < q_2 < \cdots < q_k$ be some prime numbers distinct from p_j. Given $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$, let $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$.

Lemma 2.1. *We have*

$$\frac{\sigma(n)}{n} < \prod_{\ell=1}^{k} \frac{q_\ell}{q_\ell - 1} \leq \prod_{\substack{\ell=1,\ldots,j-1 \\ j+1,\ldots,k+1}} \frac{p_\ell}{p_\ell - 1} = \frac{n}{\varphi(n)}. \tag{2.1}$$

Proof. The first inequality follows from the fact that for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$

$$\frac{\sigma(p^\alpha)}{p^\alpha} = \frac{p - \frac{1}{p^\alpha}}{p-1} \nearrow \frac{p}{p-1} \text{ as } \alpha \to \infty. \tag{2.2}$$

The second inequality follows from the fact that $p_i \leq q_i$ for all $1 \leq i \leq k$. □

Note that

$$\frac{n}{\varphi(n)} = \left(\prod_{\substack{\ell=1,\ldots,j-1 \\ j+1,\ldots,k+1}} \frac{p_\ell + 1}{p_\ell} \right) \left(\prod_{\substack{\ell=1,\ldots,j-1 \\ j+1,\ldots,k+1}} \frac{p_\ell^2}{p_\ell^2 - 1} \right) =: A(k)B(k). \tag{2.3}$$

We can bound $A(k)$ as follows:

$$\log(A(k)) = \sum_{\substack{\ell=1,\ldots,j-1 \\ j+1,\ldots,k+1}} \log\left(1 + \frac{1}{p_\ell}\right) \leq \sum_{\substack{\ell=1,\ldots,j-1 \\ j+1,\ldots,k+1}} \frac{1}{p_\ell} = \left(\sum_{\ell=1}^{k+1} \frac{1}{p_\ell} \right) - \frac{1}{p_j} \tag{2.4}$$

and

$$\sum_{\ell=1}^{k+1} \frac{1}{p_\ell} \leq \log(\log(p_{k+1})) + c_1 + \frac{5}{\log(p_{k+1})}, \tag{2.5}$$

where $c_1 \approx .261497$ by Theorem 1.10 in [Ten95]. Thus we obtain

Lemma 2.2. *For all $k \in \mathbb{N}$,*

$$A(k) \leq \log(p_{k+1}) \exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log(p_{k+1})} \right). \tag{2.6}$$

Combining [Dus99] and Theorem 3 from [RS62], we obtain the following:

Theorem 2.3. *For $k \geq 6$,*

$$k(\log(k) + \log(\log(k)) - 1) < p_k < k(\log(k) + \log(\log(k))). \tag{2.7}$$

Furthermore, combining Lemma 2.2 and Theorem 2.3, we see that

Lemma 2.4. *For $k \geq 6$, $A(k) < C(k)$ where*

$$\begin{aligned} C(k) &= \log((k+1)(\log(k+1) + \log(\log(k+1)))) \\ &\quad \exp\left(c_1 - \frac{1}{p_j} + \frac{5}{\log((k+1)(\log(k+1) + \log(\log(k+1)) - 1))} \right). \end{aligned} \tag{2.8}$$

Now, put $m = p_{k+1}\#/p_j$. Our goal is to show the following, since it implies that Robin's inequality for n as above:

Theorem 2.5. *For any $j \in \{1, 2, 3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies*

$$C(k)B(k) < e^\gamma \log(\log(m)). \tag{2.9}$$

Corollary 2.6. *Suppose Theorem 2.5 holds. Then Robin's inequality holds for n as above.*

Proof. We calculate

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq A(k)B(k) < C(k)B(k) < e^{\gamma}\log(\log(m)) \leq e^{\gamma}\log(\log(n)), \tag{2.10}$$

where the last inequality follows from the fact that $m \leq n$. □

Definition 2.7. The *Chebyshev function* is defined as follows:

$$\theta(x) = \sum_{p\in\mathbb{P}, p\leq x} \log(p) = \log\left(\prod_{p\in\mathbb{P}, p\leq x} p\right). \tag{2.11}$$

Theorem 2.8. *For $x \geq 529$,*

$$\prod_{\substack{p\in\mathbb{P}\\ p\leq x}} p = e^{\theta(x)} > e^{x\left(1-\frac{1}{2\log x}\right)} \geq (2.51)^x. \tag{2.12}$$

Proof. The first inequality is given by (3.14) in [RS62] and the second follows from computations since the function $f(x) = 1 - \frac{1}{2\log x}$ increases for $x > 1$. □

Lemma 2.9. *For $k \geq 99$,*

$$\log(\log(m)) > \log((k+1)(\log(k+1) + \log(\log(k+1)) - 1)\log(2.51) - \log(p_j)) =: D(k).$$

Proof. Noting that $k \geq 99$ implies $p_{k+1} > 529$, we calculate

$$\begin{aligned}\log(\log(m)) &= \log\left(\log\left(\frac{p_{k+1}\#}{p_j}\right)\right) > \log\left(\log\left(\frac{(2.51)^{p_{k+1}}}{p_j}\right)\right)\\ &= \log(p_{k+1}\log(2.51) - \log(p_j))\\ &> \log((k+1)(\log(k+1) + \log(\log(k+1)) - 1)\log(2.51) - \log(p_j)),\end{aligned} \tag{2.13}$$

where the last inequality uses Theorem 2.3. □

The following implies Theorem 2.5:

Proposition 2.10. *For $j \in \{1,2,3\}$, there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies*

$$C(k)B(k) < e^{\gamma}D(k). \tag{2.14}$$

Proof. Denote

$$\begin{aligned}\widetilde{C}(k) = e^{-c_1+\frac{1}{p_j}}C(k) = \log((k+1)(\log(k+1) + \log(\log(k+1))))&\\ \exp\left(\frac{5}{\log((k+1)(\log(k+1)+\log(\log(k+1))-1))}\right)&\end{aligned} \tag{2.15}$$

and

$$\widehat{C}(k) = \exp\left(\frac{5}{\log((k+1)(\log(k+1)+\log(\log(k+1))-1))}\right). \tag{2.16}$$

Multiplying both sides of (2.14) by $e^{-c_1+\frac{1}{p_j}}p_j^2/(p_j^2-1)$, we obtain

$$\widetilde{C}(k)\prod_{\ell=1}^{k+1}\frac{p_\ell^2}{p_\ell^2-1} < \frac{e^{\gamma-c_1+\frac{1}{p_j}}p_j^2}{p_j^2-1}D(k). \tag{2.17}$$

Noting that

$$\prod_{\ell=1}^{k+1} \frac{p_\ell^2}{p_\ell^2 - 1} \nearrow \frac{\pi^2}{6} \text{ as } k \to \infty, \tag{2.18}$$

we see that (2.17) is implied by

$$\widetilde{C}(k) < \frac{6p_j^2 e^{\gamma - c_1 + \frac{1}{p_j}}}{\pi^2(p_j^2 - 1)} D(k) =: E_j D(k). \tag{2.19}$$

Raising both sides to the power of e, we see that (2.19) is implied by

$$\begin{aligned} &[(k+1)(\log(k+1) + \log(\log(k+1))]^{\widehat{C}(k)} \\ &< [(k+1)(\log(k+1) + \log(\log(k+1)) - 1)\log(2.51) - \log(p_j)]^{E_j}. \end{aligned} \tag{2.20}$$

(2.20) is equivalent to

$$\begin{aligned} 1 &< [(k+1)(\log(k+1) + \log(\log(k+1)))]^{-\widehat{C}(k) + E_j} \\ &\quad \left[1 - \frac{(k+1)\log(2.51) + \log(p_j)}{(k+1)(\log(k+1) + \log(\log(k+1)))}\right]^{E_j}. \end{aligned} \tag{2.21}$$

Noting that $E_j > 1$ for $j \in \{1, 2, 3\}$, we see that there exists a $K_j \in \mathbb{N}$ such that $k \geq K_j$ implies $-\widehat{C}(k) + E_j > \epsilon$ for some $\epsilon \in (0, 1)$. If needed, we can increase K_j so that $k \geq K_j$ implies

$$\left[1 - \frac{(k+1)\log(2.51) + \log(p_j)}{(k+1)(\log(k+1) + \log(\log(k+1)))}\right]^{E_j} > \epsilon, \tag{2.22}$$

and also so that $k \geq K_j$ implies

$$1 < \epsilon[(k+1)(\log(k+1) + \log(\log(k+1))]^{\epsilon}, \tag{2.23}$$

which implies (2.21). □

2.2. **All numbers not divisible by one of the prime numbers 2,3,5.** Letting $j = 1$ in (2.19), we seek to show that

$$\widetilde{C}(k) < \frac{8e^{\gamma - c_1 + .5}}{\pi^2} D(k). \tag{2.24}$$

Lemma 2.11. *For $k \geq 13042$, $\widehat{C}(k) < 1.525$.*

Proof. $\widehat{C}(k)$ is decreasing, so the result follows from computation. □

Denote $f(k) = (k+1)(\log(k+1) + \log(\log(k+1))$. Applying Lemma 2.11to (2.24) and performing some algebraic manipulations, our goal reduces to showing that

$$\log(f(k)) < \frac{8e^{\gamma - c_1 + .5}}{\pi^2(1.525)} \log((f(k) - 1)\log(2.51) - \log(2)). \tag{2.25}$$

Raising both sides to the power of e, this becomes

$$1 < f(k)^{2.0166} \left[1 - \frac{(k+1)\log(2.51) - \log(2)}{f(k)}\right]^{1.20166}. \tag{2.26}$$

The RHS of (2.26) is increasing, and a computation reveals that it holds for $k \geq 13042$. Additionally, using Lemma 2.1, one can check that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} < e^{\gamma} \log(\log(m)) \tag{2.27}$$

for $k \geq 3$. Finally, when $k \in \{1, 2\}$, we check that

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)} \leq \frac{15}{8} < e^{\gamma} \log(\log(n)) \tag{2.28}$$

for $n \geq 680$. This confirms the following for $j = 1$:

Theorem 2.12. *For $j \in \{1, 2, 3\}$, Robin's inequality holds for every natural number > 5040 which is not divisible by p_j.*

To confirm Theorem 2.12 when $j \in \{2, 3\}$, one can repeat the above process to see that, for sufficiently big k, (2.19) is satisfied. The cases with smaller k have been verified in [MP20].

2.3. **Primorials and sufficiently big even numbers.** Fix $k \in \mathbb{N}$ and let n be odd. We consider numbers of the form $2^k n$. We calculate

$$\frac{\sigma(2^k n)}{2^k n} = \frac{\sigma(2^k)}{2^k} \frac{\sigma(n)}{n} < \frac{\sigma(2^k)}{2^k} \frac{n}{\varphi(n)} = \frac{\sigma(2^k)}{2^k} \frac{\varphi(2^k)}{2^k} \frac{2^k n}{\varphi(2^k n)} = \left(1 - \frac{1}{2^{k+1}}\right) \frac{2^k n}{\varphi(2^k n)}. \tag{2.29}$$

Applying Theorem 15 from [RS62], we know

$$\left(1 - \frac{1}{2^{k+1}}\right) \frac{2^k n}{\varphi(2^k n)} < \left(1 - \frac{1}{2^{k+1}}\right) \left(e^{\gamma} \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right). \tag{2.30}$$

We ask which n satisfy

$$\left(1 - \frac{1}{2^{k+1}}\right) \left(e^{\gamma} \log(\log(2^k n)) + \frac{2.51}{\log(\log(2^k n))}\right) < e^{\gamma} (\log(\log(2^k n))). \tag{2.31}$$

This is equivalent to asking when

$$\frac{2.51(2^{k+1} - 1)}{e^{\gamma}} < (\log(\log(2^k n)))^2 \tag{2.32}$$

holds, which is when

$$n > \frac{e^{e^{\sqrt{\frac{2.51(2^{k+1}-1)}{e^{\gamma}}}}}}{2^k} =: b(k). \tag{2.33}$$

Thus, we obtain the following:

Theorem 2.13. *Given any $k \in \mathbb{N}$, Robin's inequality holds for all numbers of the form $2^k n$ when n is odd and satisfies* (2.33)*.*

In particular, we have the following:

Corollary 2.14. *If $n \geq 620$ is odd, then Robin's inequality holds for $2n$. Furthermore, Robin's inequality holds for all primorials > 30.*

Proof. The first statement follows immediately from Theorem 2.13 and the second follows from the computation of primorials < 1240.3 □

2.4. **All 21-free numbers.** The results of the previous subsection are based on the inequality in Theorem 15 from [RS62]. This inequality can be improved by using a sharper bound stated in [AN22]:

$$\frac{m}{\varphi(m)} < e^{\gamma}\log(\log(m)) + \frac{.0168}{(\log(\log(m)))^2} \tag{2.34}$$

for $m \geq 10^{10^{13.11485}} =: C$. Using the same reasoning as in the proof of Theorem 2.13, we derive the following result:

Theorem 2.15. *Given $k \in \mathbb{N}$, Robin's inequality holds for all numbers of the form $2^k n$ when n is odd and satisfies*

$$n > \frac{e^{e^{\sqrt[3]{\frac{.0168(2^{k+1}-1)}{e^{\gamma}}}}}}{2^k} =: \tilde{b}(k). \tag{2.35}$$

We can thus conclude the following:

Theorem 2.16. *Robin's inequality holds for all 21-free numbers.*

Proof. Let $k \in \mathbb{N}$ and n be an odd natural number. If $n > \tilde{b}(k)$, n satisfies Robin's inequality by Theorem 2.15. If not, note that n satisfies Robin's inequality if $5040 < 2^k n \leq 2^k \tilde{b}(k) \leq C$ by Theorem 13 in [MP20]. Recalling the definition of ℓ-free numbers, we see that if $2^k \tilde{b}(k) < C$, then all $(k+1)$-free numbers satify Robin's inequality. Indeed, setting $k = 20$, we calculate

$$\log(2^{20}\tilde{b}(20)) < 6(10^{11}) < 2.3(10^{13.11485}) < \log(C). \qquad \square$$

2.5. **Almost every number.**

Definition 2.17. The *natural density* of a set E is

$$d(E) = \lim_{n\to\infty} \frac{\#E \cap \{1,2,\ldots,n\}}{n} \tag{2.36}$$

when the limit exists.

Theorem 2.18. *Denote by $\mathcal{R}$ the set of numbers satisfying Robin's inequality. Then the natural density of $\mathcal{R}$ is 1.*

Proof. We will prove that the natural density of $\mathcal{R}^c$ is 0. Fix $\epsilon > 0$. Let $E_k = \{2^k n : n \in \mathbb{N}_{\text{odd}}, n \leq b(k)\}$ and note that $\mathcal{R}^c \subseteq \bigcup_{k\geq 1} E_k$ by Theorem 2.12 and Theorem 2.13.[1] Pick M so that $\sum_{k=M+1}^{\infty} \frac{1}{2^k} < \frac{\epsilon}{2}$. For $n \in \mathbb{N}$ we calculate

$$\begin{aligned} \frac{\#\mathcal{R}^c \cap \{1,2,\ldots,n\}}{n} &\leq \frac{\#\bigcup_{k\geq 1} E_k \cap \{1,2,\ldots,n\}}{n} = \frac{\sum_{k\geq 1} E_k \cap \{1,2,\ldots,n\}}{n} \\ &= \frac{\sum_{k=1}^{M} \#E_k \cap \{1,2,\ldots,n\} + \sum_{k=M+1}^{\infty} \#E_k \cap \{1,2,\ldots,n\}}{n}, \end{aligned} \tag{2.37}$$

where the first equality follows from the fact that the E_k's are disjoint. Noting that $\sum_{k=1}^{M} \#E_k \cap \{1,2,\ldots,n\} < \infty$ for all $n \in \mathbb{N}$, we see that we can pick N so that $n \geq N$ implies that the RHS of (2.37) is $< \epsilon$, completing our proof. $\square$

[1]Here $\mathbb{N}_{\text{odd}} := \{1,3,\ldots\}$.

3. The Lagarias and Kaneko-Lagarias Inequalities

3.1. **Superabundant numbers.** Let $\Gamma(x)$ denote the gamma function. We define two functions:

$$H(x) = \int_0^1 \frac{t^x - 1}{t - 1}\,\mathrm{d}t, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{3.1}$$

ψ is known as the *digamma function*. One can verify that H is smooth for $x \geq 1$ and that $H(n) = H_n$ for all $n \in \mathbb{N}$. It's also easy to see that ψ, known as the digamma function, satisfies

$$H(x) = \psi(x+1) + \gamma. \tag{3.2}$$

Lemma 3.1. *For all $x \geq 1$,*

$$H(x) < \log(x) + \gamma + \frac{1}{2x}. \tag{3.3}$$

Proof. By (2.2) from [Alz97],

$$\psi(x) < \log(x) - \frac{1}{2x} \tag{3.4}$$

for all $x \geq 1$. Then we use (3.2) and $\psi(x+1) = \psi(x) + \frac{1}{x}$ to finish. □

Lemma 3.2. *For all $x \geq 4$,*

$$H(x) < \frac{2\log(x)}{1 + \frac{6}{\pi^2 x}}. \tag{3.5}$$

Proof. By Lemma 3.1, it suffices to show that

$$\log(x) + \gamma + \frac{1}{2x} < \frac{2\log(x)}{1 + \frac{6}{\pi^2 x}} \tag{3.6}$$

for $x \geq 4$. By arithmetic manipulations, (3.6) becomes

$$\frac{1}{\pi^2 x - 6}\left(\gamma\pi^2 x + \frac{\pi^2}{2} + 6\gamma + \frac{3}{x}\right) < \log(x). \tag{3.7}$$

Computation reveals that (3.7) holds for $x = 4$, and the LHS of (3.7) is decreasing while the RHS is increasing so we obtain the result. □

Lemma 3.3. *The following hold:*
- *(a) For all $n > 1$, $H_{n+1} \leq \frac{n}{\log(n)}$.*
- *(b) For all $x \geq 4$, $\log(H(x)) \leq \frac{x}{2\log(x)}$.*

Proof. (a) We can manually verify the inequality for $n \leq 6$. Noting that

$$H_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} \leq 1 + \int_1^{n+1} \frac{\mathrm{d}t}{t} = 1 + \log(n+1), \tag{3.8}$$

it suffices to show that

$$\log(x)(\log(x+1) + 1) \leq x. \tag{3.9}$$

Put $g(t) = e^t - t^2 - t - 1$. We see that $g(2) > 0$ and that $g'(t) = e^t - 2t - 1 > 0$ for $t \geq 2$, so $g(t) > 0$ for $t \geq 2$. For $x \geq e^2 - 1$ we have

$$0 < g(\log(x+1)) = x+1-(\log(x+1))^2 - \log(x+1) - 1 < x - \log(x)(\log(x+1)+1). \tag{3.10}$$

(b) For $x \geq 4$, note that the function mapping $x \mapsto \frac{x}{\log(x)}$ is increasing. If $n \leq x < n+1$, then

$$H_n \leq H(x) < H_{n+1} \leq \frac{n}{\log(n)} \leq \frac{x}{\log(x)}. \tag{3.11}$$

For $y > 2$ we see that $\log(y) < \frac{y}{2}$, so let $y = H(x)$ and apply (3.11) finish. □

Lemma 3.4. *For $x \geq 4$,*

$$H(x)\log(H(x)) < \frac{x^2}{x + \frac{6}{\pi^2}}. \tag{3.12}$$

Proof. Apply Lemma 3.1 and Lemma 3.3. □

Lemma 3.5. *For $x \geq 4$,*

$$H'(x) > \frac{H(x)\log(H(x))}{x^2}. \tag{3.13}$$

Proof. We will use (51) from [FD14] which states that

$$\frac{1}{\psi'(x)} \leq x + \frac{6}{\pi^2} - 1 \tag{3.14}$$

for $x \geq 1$. We calculate

$$H'(x) = \psi'(x+1) \geq \frac{1}{x + 6\pi^2} > \frac{H(x)\log(H(x))}{x^2}, \tag{3.15}$$

where the equality follows from taking the derivative of (3.2) and the second inequality follows from Lemma 3.4. □

Proposition 3.6. *The function*

$$g(x) = \frac{\exp(H(x))\log(H(x))}{x} \tag{3.16}$$

is increasing for $x \geq 4$.

Proof. We start with (3.5) from [Lag02]:

$$\begin{aligned} H_n &= \log(n) + \gamma + \int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x \\ \implies \exp(H_n) &= e^\gamma n \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x\right) \\ \implies \frac{\exp(H_n)\log(H_n)}{n} &= e^\gamma \log(H_n) \exp\left(\int_n^\infty \frac{x - \lfloor x \rfloor}{x^2}\,\mathrm{d}x\right). \end{aligned} \tag{3.17}$$

Given $k \in \mathbb{N}$, put

$$g_k(x) = e^\gamma \log(H(x)) \exp\left(\int_x^k \frac{t - \lfloor t \rfloor}{t^2}\,\mathrm{d}t\right) \tag{3.18}$$

so that $\lim_{k\to\infty} g_k(x) = g(x)$. We compute

$$g_k'(x) = e^\gamma \exp\left(\int_x^k \frac{t-\lfloor t\rfloor}{t^2}\,\mathrm{d}t\right)\left(\frac{H'(x)}{H(x)} + \log(H(x))\left(-\frac{x-\lfloor x\rfloor}{x^2}\right)\right), \tag{3.19}$$

so $g_k'(x) > 0$ if and only if

$$\frac{H'(x)}{H(x)} + \log(H(x))\left(-\frac{x-\lfloor x\rfloor}{x^2}\right) \geq \frac{H'(x)}{H(x)} - \frac{\log(H(x))}{x^2} > 0, \tag{3.20}$$

which is the content of Lemma 3.5. Thus, $g(x)$ is the limit of monotonically increasing functions and is therefore monotonically increasing. □

Corollary 3.7. *The sequence*

$$\left\{\frac{\exp(H_n)\log(H_n)}{n}\right\}_{n=1}^{\infty} \tag{3.21}$$

is monotonically increasing.

Proof. Proposition 3.6 gives the result for $n \geq 4$ and we can manually check the smaller cases. □

Definition 3.8. A number n is *superabundant* if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.

Theorem 3.9. *If there are counterexamples to the Kaneko-Lagarias inequality, the smallest such counterexample is a superabundant number.*

Proof. Suppose, for sake of contradiction, that m is the smallest counterexample to the Kaneko-Lagarias inequality and that m is not superabundant. Let n be the greatest superabundant number $< m$. We calculate,

$$\frac{\sigma(n)}{n} > \frac{\sigma(m)}{n} \geq \frac{\exp(H_m)\log(H_m)}{m} > \frac{\exp(H_n)\log(H_n)}{n}, \tag{3.22}$$

so $n < m$ violates the Kaneko-Lagarias inequality: a contradiction. □

3.2. **Connection to Robin's inequality.**

Theorem 3.10. *If Robin's inequality holds for some $n \in \mathbb{N}$, then the Kaneko-Lagarias inequality holds for n.*

Proof. We use the approximation

$$H_n \geq \log(n) + \gamma + \frac{1}{2n+1} \tag{3.23}$$

to calculate

$$\frac{\exp(H_n)\log(H_n)}{n} \geq \frac{e^{\gamma+\frac{1}{2n+1}}\, n \log\left(\log(n)+\gamma+\frac{1}{2n+1}\right)}{n} > e^\gamma \log(\log(n)), \tag{3.24}$$

which implies the result. □

Note that we obtain the same result for the Lagarias inequality.

Acknowledgments

The first author thanks Jeff Lagarias for his comments and the references he provided. The third author thanks Keith Briggs for providing his code used to compute superabundant numbers, Perry Thompson and Owen McAllister for their help implementing it in Rust, and Jean-Louis Nicolas for sharing the paper [AN22]. We also thank the anonymous referees for their helpful comments, which certainly improved the quality of this paper.

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